Wealth Effect with CARA Utility: Portfolio Choice in the Presence of Costly Information

Dmitry S. Makarov∗
Astrid V. Schornick†

May 31, 2007

This paper has benefited from conversations with Suleyman Basak, Fransisco Gomes, Anna Pavlova, Raman Uppal. All errors are ours.

∗dmakarov@london.edu. London Business School, Regent’s Park, NW1 4SA, London, United Kingdom.
†aschornick@london.edu. London Business School, Regent’s Park, NW1 4SA, London, United Kingdom.
Abstract

A well-known criticism of portfolio choice models with CARA utility is the absence of wealth effect - the dollar amount invested into risky assets does not depend on wealth. We propose a simple and tractable way of incorporating the wealth effect by making the absolute risk aversion parameter wealth-dependent. We then apply our idea to two economic settings that look at the impact of information acquisition on portfolio choice. First, in a setting with Bayesian investors, we demonstrate that our method leads to a simple, tractable analysis as opposed to the alternative approach of using CRRA utility and resorting to approximate solution. Second, we present a model with ambiguity-averse investors and show that the model explains several observed patterns of households’ stockholding: a) non-participation of a large fraction of households, with poorer ones more likely to stay away from the stock market, b) wealth share invested into risky assets increases with wealth, c) market participation increases over time, while the equity premium decreases.
1 Introduction

When studying the impact of costly information acquisition in financial markets, the question of wealth distribution arises. It seems to be a pervasive phenomenon across asset markets that people choose to spend very different amounts on information regarding investment opportunities and also have very different investment policies – wealthier people seem to be better informed and also invest more in risky assets.

The natural question to follow from this is why non-participants do not simply purchase information, so as to be better able to participate in the markets and identify good investment opportunities. Therefore it is important to study the interaction of wealth with decisions of information acquisition as well as investment in a general equilibrium environment, so as to increase understanding how these effects work together to bring about such a dominant separation between those who choose to invest and those who do not.

Generally, issues of information acquisition or informational advantage are often studied assuming returns are normally distributed and investors have exponential utility. Such utility specification implies that investors have constant absolute risk aversion. Important early papers in this area include (Grossman & Stiglitz 1980), (Hellwig 1980), (Diamond & Verrecchia 1981) and (Verrecchia 1982), among others.

The tractability of this setup comes at a cost however. Assuming constant absolute risk aversion on behalf of all agents severely restricts the study of the cross section of investors and how their decisions interact and impact one another. Specifically, no matter what the level of wealth is, all agents with the same level of absolute risk aversion will invest exactly identical dollar amounts into a stock. This leads to the counterfactual implication that the fraction of wealth invested into risky assets decreases with wealth, while various studies find that wealthier households in fact invest a larger fraction of their financial wealth into stocks than poorer ones.

To make things worse for CARA-normal models, (Bernardo & Judd 2000) showed that it is not only portfolio shares that are sensitive to the utility specification. They solved numerically

\[ \text{\footnotesize \cite{Bernardo2000} \text{\footnotesize Apart from constant absolute risk aversion, other assumptions have also been found to be crucial for models' qualitative results. For example, (Barlevy & Veronesi 2000) follow a Grossman & Stiglitz framework, but assume that the risky payoff is binomially, rather than normally, distributed. They show that under this assumption the prices may not become more informative as more investors acquire information – in contrast to Grossman & Stiglitz result. In this paper we focus only on the (absence of) a wealth effect.}} \]
a variation of the (Grossman & Stiglitz 1980) model in which investors have CRRA preferences, and found that the model predictions about the general equilibrium effects, e.g. regarding the price informativeness, are not robust once more realistic preferences are introduced. Given their results, the authors go as far as claiming that “exponential utility implies that a trader’s holding of risky assets is unrelated to his wealth, an unreasonable assumption making dynamic, general equilibrium extensions (De Long et al., 1990,1991; Wang, 1994) of this model unrealistic.”

Of course, if wealth were distributed evenly across the population or wealth had been shown to be irrelevant for investment decisions, this would not be an important issue. However, it is well known that in most countries with developed asset markets there is a great deal of wealth heterogeneity across the population. Moreover, people who are wealthier tend to invest more into the stock market than poorer people, who tend to save less overall and hold fewer risky assets. A large proportion of households does not participate in the stock market at all, and (Guiso, Haliassos & Jappelli 2003) as well as (Vissing-Jorgensen 2003) find that participation is positively correlated with wealth. So wealth seems to be an important factor for understanding the observed patterns of stockholdings. Existing models with CARA utility have little to say about how wealth heterogeneity affects portfolio choice, stock prices, volatility, and other important parameters.

Despite all the criticism, models with CARA utility and normally distributed returns remain the benchmark modelling device whenever a noisy rational expectations setting is considered. Recent examples include (Llorente, Michaely, Saar & Wang 2002), (Allen, Morris & Shin forthcoming), (Makarov & Rytchkov 2006). The profession seems to have accepted the above-described limitations as a reasonable price to pay for the resulting analytical tractability. As summarized by (Admati 1989)

For tractability, all of the models of noisy rational expectations equilibrium ... assume exponential utility functions and normal distributions. Although these models capture many important phenomena, this limitation should be noted. Needless to say, tractable [our emphasis] models with different parametric assumptions are sorely needed.

In other words, various examples highlighting the shortcomings of CARA-normal models are
unlikely to make researchers abandon the CARA-normal setting unless a tractable alternative is available.

In this paper we propose a slight reformulation of the CARA utility function, allowing us to incorporate the wealth effect into models with CARA utility, thus addressing one of the main criticisms of these preferences. The idea is rather simple: to stay within the CARA-normal framework, but make the absolute risk tolerance of each investor depend on her initial endowment, in a way consistent with the empirical evidence. While the existing evidence is definitive that absolute risk aversion decreases with wealth, the evidence on how relative risk aversion changes with wealth is somewhat mixed. Some papers find that relative risk aversion increases with wealth, others find the opposite. In this paper we assume that the relative risk aversion is constant which seems to be a reasonable compromise. We incorporate the wealth effect in such a way that the new parametrization of the CARA utility function resembles CRRA utility in terms of the relation between wealth, absolute risk aversion and resulting investment policy. Specifically, we set the absolute risk tolerance of a CARA investor to be an increasing linear function of her current wealth.

The economic interpretation is as follows. We envision investors whose true preferences are CARA. However, when deciding on investments into a risky asset, an investor evaluates it using the absolute risk aversion that depends on her current – pre-decision – level of wealth. With CRRA utility, such scaling of the absolute risk aversion as wealth changes happens automatically. Our approach is just an alternative way to capture exactly the same phenomenon – rich and poor people evaluate the same gamble differently. Which of these two – CARA with wealth-dependent risk aversion or CRRA – approaches lends more tractability depends mainly on the economic setting. For example, CRRA utility is known to work well in continuous-time models, at least in setups dealing solely with terminal consumption of investors. However, as we show in the subsequent analysis, in various discrete-time settings, implementing our method leads to a tractable analysis, while one would need to use either numerical or approximation

\[ \text{\textsuperscript{2}} \text{See (Peress 2004) for a review of the evidence.} \]

\[ \text{\textsuperscript{3}} \text{In this paper we discuss only two utility specifications: CARA and CRRA. In doing so, we abstract from the entire branch of recent literature on the new utility functions (e.g., internal and external habit formation, (Epstein & Zin 1989), (Kahneman & Tversky 1979)). The ability of these types of utility functions to explain various phenomena stand in contrast to their analytical intractability, often leading to the numerical simulations being required.} \]
techniques to solve the corresponding model with CRRA investors.

While the idea is simple, it has been largely overlooked by the existing literature. The general perception seems to be that the only way to account for the wealth effect is to introduce other preferences, different from CARA. We are aware of only one study that uses a reasoning which is somewhat similar to ours. (Broner, Gelos & Reinhart 2006) consider a CARA-framework in which absolute risk aversion depends on investors’ past performance. They refer to CRRA utility as being “more realistic”, while allowing absolute risk aversion to change with the performance is presented as a trick that allows them to solve the model in closed-form. However, the existing empirical evidence on investors’ attitude to risk and how it depends on his wealth does not allow one to distinguish between the specifications of CRRA versus CARA with wealth-dependent risk aversion. In this respect, we argue that the latter could be a valid representation of the way investors assess risk. Methodologically, one of the goals of this paper is convincing the reader to reassess the notorious feature – “CARA utility implies no wealth effect.” Broner et al. do not make this point in general.

Having discussed the methodology itself, we now turn to the paricular economic questions that we can address thereby. Our main focus is explaining theoretically several well-documented patterns of stockholding across households. First, a significant proportion of households does not participate in the stock market at all, as shown by (Mankiw & Zeldes 1991). It is documented that the poorer households are more likely to refrain completely from entering the stock market. Second, stock market participation has been increasing over time. Third, looking at the households that do participate, the wealthier ones invest a larger share of their wealth into risky securities than the poorer ones.

4Note that one remains an exponential function and the other a power function, so while the investment policies will look the same, this is not the case for all characteristics based on the shape of the utility function.

5(Peress 2004) and (Bernardo & Judd 2000) are two recent examples of papers that criticize CARA preferences for the absence of wealth effect, and then introduce the new preferences with decreasing absolute risk aversion. As a result of deviating from CARA-normal setting, both papers can no longer solve the model in closed-form and have to resort to log-linearization as in Peress or numerical techniques as in Bernardo & Judd. In a recent paper (Cao, Wang & Zhang 2005) investigate the portfolio choice of heterogeneous uncertainty averse investors. They say that “[due to CARA utility] the initial endowment of an investor does not affect his optimal portfolio holding. With other utilities, however, this is often not the case. One obvious example is CRRA utility. ...[Abstracting away the endowment effect] gives us the analytic tractability which greatly facilitates the analysis.” So Cao et al. suggest, similarly to the above two papers, that to capture the wealth effect one needs to abandon the CARA framework. In this paper we demonstrate that our approach works well and leads to novel economic predictions both in a setting with Bayesian investors (similar to Peress and Bernardo & Judd) and ambiguity-averse investors (similar to Cao et al.).
We first consider a model with Bayesian investors. Our main goal is to demonstrate the advantage of staying within the CARA-normal setting and making the absolute risk aversion wealth dependent. Indeed, our analysis in this case is a straightforward extension of (Verrecchia 1982). The alternative path of changing preferences to CRRA leads to the tractability breaking down. One then has to resort to log-linearization to arrive at the solution. Importantly, our analytical solution exemplifies the pitfalls of log-linearization in this economic setup. (Peress 2004) shows that under log-linearized CRRA preferences, the average share of wealth invested into the risky asset increases with wealth. We show that this relationship is ambiguous and depends on the model’s parameters. Log-linearization, being a non-affine transformation, distorts the correlation between (random) equilibrium demand and price, thus leading to the putative unambiguously positive relationship between amount of information purchased and money invested. We explicitly present the parameter conditions necessary for this correlation effect to lead to higher wealth levels being associated with lower fractions of wealth invested into the stock market. In this paper we do not calibrate the model to evaluate the magnitude of the correlation term. In fact, one may think that empirical studies would support the idea of a positive correlation between wealth level and fractions invested. But looking at the tractable method introduced in this paper suggests that the concept of information acquisition may not be a clear explanation for this observation.

While the model with Bayesian investors can potentially explain why wealthy investors put relatively more of their wealth into risky assets, it cannot shed any light on questions related to non-participation, due to the strict continuity of investment decisions. To study this we turn to a model that builds on the work of (Cao et al. 2005). They consider a general equilibrium model with ambiguity-averse investors. Investors are assumed to differ in their levels of uncertainty about the true expected risky stock return. We extend this work along two dimensions.

First, we incorporate the process of information acquisition in reduced form by relating the investors’s wealth to her level of uncertainty, whereby wealthier households have a narrower uncertainty interval around the true mean than poorer ones. The reason is that wealthier investors choose to purchase more information, thus making their information set more precise. To focus on the main economic questions, we do not model explicitly the optimal information acquisition.
Second, in line with our earlier discussion, we set the absolute risk tolerance of an investor to be proportional to her wealth. To sum up, the distinctive features of our model, as compared to (Cao et al. 2005), is that wealthier households are less risk averse (absolutely) than poorer ones and also have more precise information about the true mean of the stock return.

Our model can generate all three of the above-mentioned patterns of stockholding: non-participation and its correlation with household wealth, increase in participation over time, and wealth share invested into risky assets increasing with household wealth.

Introducing the wealth effect into CARA utility leads not only to novel economic results but also helps to realign some of Cao et al.’s original conclusions with those of other models in the literature. In Cao et al., when market participation decreases, the market risk premium decreases as well. In other words, their model with endogenous restricted participation makes the equity premium puzzle (as first noted by (Mehra & Prescott 1985)) even worse compared to the full-participation case. This is a surprising result as it stands in contrast to the conclusion of some models with exogenous limited participation, e.g. (Basak & Cuoco 1998), who argue that limited participation can help resolve the puzzle. In our analysis we find support for Basak and Cuoco’s result by showing that the underlying reason for this difference is the presence of a wealth effect, rather than the presence of ambiguity. Once the wealth effect is incorporated into the setting of Cao et al., a decrease in market participation leads to an increase in the equity premium.

Some other results of the base model also no longer hold once the “wealth-adjusted” preferences are introduced. For example, in a full-participation case Cao et al. find that investors’ uncertainty dispersion has no effect on equilibrium price – only average uncertainty matters. In our setting both average uncertainty and the dispersion affect the price. To sum up, given the significant differences between our results and those of Cao et al., we argue, similarly to (Bernardo & Judd 2000), that conclusions of CARA-models need to be studied to assess how general their results are.

The main limitation of our approach is that it cannot work in a multi-period portfolio choice setting. However, as many models of the particular type that we consider to be the main “target” for our method are also static, so they can be easily compared at least against one another. Indeed, even in many multi-period noisy rational expectation models the portfolio
The same is true for several recent papers investigating portfolio choice under ambiguity and relying on the CARA-normal assumptions: (Kogan & Wang 2002), (Cao et al. 2005) and (Garlappi, Uppal & Wang 2007) are for example also static.\footnote{See (Brunnermeier 2001) for an excellent literature review.}

However, given that static portfolio choice on the part of investors is essential for our method to work, the question is how restrictive this assumption is. Theoretically, whether the hedging component in the optimal portfolio is quantitatively significant or not depends on the economic setting. Hedging demands are found to be quantitatively important in models with predictable variations in interest rates and equity premia\footnote{While not investigating this formally, it seems straightforward that our approach can be applied in some multi-period settings provided that investors are myopic.} In other contexts, for example in (Chacko & Viceira 2005) as well as (Bacchetta & van Wincoop 2005), the hedging term was found to be small. There are other cases when assuming that investors are myopic is justified. For example, if the main focus of a model is not on the portfolio choice but other aspects like information aggregation, liquidity or higher-order expectations, it can be reasonable to disregard hedging demands, so as to focus on the main economic questions.

Interestingly, while any model in which investors are myopic usually has to defend this assumption by referring to considerations of tractability, it is not at all obvious that the real market participants are non-myopic. In a recent review, (Brandt 2005) writes: “The myopic portfolio choice is an important special case for practitioners and academics alike. There are, to my knowledge, few financial institutions that implement multi-period investment strategies involving hedging demands...A common justification from practitioners is that the expected utility loss from errors that could creep into the solution of a complicated dynamic optimization problem outweighs the expected utility gain from investing optimally as opposed to myopically.”

Thus, it may well be the case that myopia on the part of investors is not just a simplifying assumption but rather a realistic description of the way investors form their portfolios.

While this paper does not answer the question whether investors are indeed myopic, it makes contributions along two dimensions. First, on the methodological front, we propose a simple way of incorporating the wealth effect into existing CARA-normal models when portfolio choice is static (or myopic). Compared to using the generally more favored CRRA utility to describe

\footnote{See (Campbell & Viceira 2002) for a summary of the research.}
investor behavior, we demonstrate that our approach a) is much simpler to implement; b) results in a closed-form solution; c) leads to similar conclusions as the model with CRRA before the log-linear approximation is employed; d) highlights an intrinsic weakness of the non-affine log-linear transformation for studying portfolio choice problems.

Second, in terms of economic implications, we present a model that explains several salient features of household stockholding. While there exist models that explain each of these features individually, as far as we know our paper is the first to provide a joint explanation.

The rest of the paper is organized as follows. Section 3.2 provides a simple example that demonstrates the intuition behind our idea, and also discusses the scope of its applicability. Section 3.3 incorporates the wealth effect in a Bayesian setting of (Verrecchia 1982). Section 3.4 incorporates the wealth effect in a multi-prior setting of (Cao et al. 2005). Section 3.5 concludes. The appendix contains all proofs.

2 Motivation

Before presenting the main model, we consider an example that illustrates our idea in the simplest possible setting. We look at the portfolio choice problem in a one-period setting for two utility specifications: constant absolute and constant relative risk aversion – CARA and CRRA.

There are two assets in the economy: a riskless bond with (gross) return 1 and a risky stock with payoff \( \tilde{u} \), distributed normally with \( N(\mu_0, \sigma_0^2) \). First, consider the portfolio choice problem of an investor with the CARA utility function over wealth \( w \):

\[
- \exp(-w/r).
\]

Here \( r > 0 \) is the investor’s absolute risk tolerance (the reciprocal of the absolute risk aversion). The solution to this optimization problem for CARA utility is well-known and thus given without derivation. In the optimal portfolio, the dollar amount invested into the risky stock, \( d \), is equal to

\[
d = r\frac{\mu_0 - 1}{\sigma_0^2}. \tag{1}
\]
This equation demonstrates the problematic characteristic of CARA utility specification. A CARA investor invests the same amount of money into the risky stock regardless how rich he is. Accordingly, this means that wealthier households invest a smaller fraction of their wealth into the stock market than poorer ones. As documented by many empirical studies, this is a counterfactual implication.

Now, suppose the preferences of an agent are given by a CRRA utility function

$$w^{1-a} (1-a), \ a > 0,$$

where $a$ is the coefficient of the relative risk aversion. In this case it is easy to show that the fraction of wealth invested into the stock is constant across levels of wealth. Indeed, denoting by $w_0$ the initial wealth, by $\theta$ the fraction of wealth invested into the stock, we get that the terminal wealth is $w = w_0(\theta(\tilde{u} - 1) + 1)$. Since $w_0$ enters the expression multiplicatively and the utility function has a power form, the optimal $\theta$ does not depend on $w_0$. Accordingly, the dollar amount invested, $\theta w_0$, is linearly increasing in the initial wealth.

For the above CRRA utility, the coefficient of absolute risk tolerance is equal to $w/a$.\footnote{By definition, for a utility function $U(w)$ the absolute risk tolerance is equal to $-U'(w)/U''(w)$.} The motivation for our approach comes from the observation that using this value in (1) in place of $r$ yields that the share of wealth invested into the risky stock becomes constant:

$$\frac{d}{w_0} = \frac{\mu_0 - 1}{a \sigma_0^2}.$$

In other words, instead of using a CRRA utility, one can get similar implications for the portfolio choice simply by making the absolute risk tolerance parameter in a CARA utility setting wealth-dependent in this manner.

A word of caution is in order. In CARA-normal models, investment policies may allow wealth to become negative. If we were to apply the above formula as it is, this would require setting the absolute risk tolerance to a negative value. However, the resulting utility would be decreasing in wealth. One way to address this problem is to ensure that when calibrating the model to match the data, the model’s parameters will ensure that scenarios with negative
wealth have a very low probability and hence can be ignored. Alternatively, we can assume that the risk tolerance can not be below a small number \( r_{\text{min}} > 0 \). This means that all investors with initial wealth leading to a risk tolerance less than \( r_{\text{min}} \) have the same “almost-zero” level \( r_{\text{min}} \).

This example illustrates the implementation of this idea in the simplest possible setting. The real test is whether (and how) it works in a more realistic, richer framework. We address this question in Sections 3.3 and 3.4.

The main limitation of our approach is that it cannot be applied to models with multi-period portfolio choice. The reason is that as wealth changes over time, the absolute risk tolerance parameter will also reflect this change. However, changing \( r \) in the utility function \( -\exp(-w/r) \) affects not only its curvature, but also the value of the utility function itself. A non-myopic investor would take all such anticipated future changes into account, which would be reflected in his portfolio. This would severely impede the tractability of the model.

To better understand why it is a problem, let us consider a simple two-period example analyzed by (Cao et al. 2005) in one of the extensions of their main model. Portfolio choice happens at \( t = 0 \) and \( t = 1 \). As usual, we solve backwards. Once we find the optimal portfolio at \( t = 1 \), we evaluate the expected utility function at the optimum to get the indirect utility function at \( t = 1 \), denoted by \( J(w_1, r_1) \) where \( w_1 \) and \( r_1 \) are the time-1 wealth and absolute risk tolerance, respectively. As shown by Cao et al.:

\[
J(w_1, r_1) = -\exp(-w_1/r_1 - f(1/r_1)),
\]

where \( f(\cdot) \) is a quadratic function. Equation (2) presents a well-known result – in a CARA-normal setting, the indirect utility function has a similar functional form to the underlying preferences up to a multiplicative factor. In our case the factor is \( \exp(f(1/r_1)) \).

The optimization problem at time \( t=0 \) is maximizing the expected value of \( J(w_1, r_1) \). Applying our approach, we would need to make \( r_1 \) proportional to \( w_1 \). But this means that \( w_1/r_1 \) term is constant and so \( J(\cdot) \) does not depend on the time-1 wealth. \( J(\cdot) \) depends on \( w_1 \) only indirectly, through the \( f(1/r_1) \) term\(^{10}\). Therefore we see, we cannot vary the absolute risk

\(^{10}\) Notice that in the increasing region of the parabola \( f(1/r_1) \), \( J(\cdot) \) does not satisfy one of the main axioms:
tolerance within the same optimization problem because the resulting scaling makes the utility function inappropriate for ranking the outcomes.

The above analysis does not imply that our approach cannot work in multi-period models. To avoid the problems with scaling of the preferences, one has to assume that investors are myopic in the sense that they use their current level of absolute risk tolerance when solving for optimal portfolios in all future periods. It is only when a future period is realized that they reset the risk tolerance. In other words, investors change their attitude to risk only when their actual wealth changes.

3 Information and Portfolio Choice: a Bayesian Approach

3.1 Economic Setting

The setting presented in this Section is similar to that in (Verrecchia 1982). We first present this original setup and then discuss the modifications.

There are two assets in the market: a risky stock and a riskless bond. The timeline is as follows. At $t = 0$ traders are endowed with assets. Given her endowment, each trader decides how much information to buy. At $t = 1$ the information is revealed and the trade among investors takes place (but no consumption). At $t = 2$ the returns on assets are realized and the traders consume their terminal wealth.

The bond serves as a numeraire and returns 1 at $t = 2$. The realized return on the risky asset, denoted by $\tilde{u}$, is not known to traders until $t = 2$. We assume that investors share the common prior belief that $\tilde{u}$ is normally distributed with mean $\mu_0$ and variance $\sigma_0^2$. Denote by $h_0$ the precision of $\tilde{u}$:

$$h_0 = \frac{1}{\sigma_0^2}. \quad (3)$$

Each investor $i$ is able to buy a signal $\tilde{y}_i$ about the true realization of $\tilde{u}$. This decision to purchase is made at $t = 0$ after observing the endowment, and the information becomes available in time for investors to make their portfolio decisions at $t = 1$. Specifically, we assume

"more is preferred to less". It is straightforward to see, given that $r_1$ linearly increases with $w_1$, that $J(w_1, r_1(w_1))$ decreases with $w_1$. 

13
that
\[ \tilde{y}_i = \bar{u} + \tilde{\epsilon}_i, \]
where \( \tilde{\epsilon}_i \) is normally distributed with mean 0 and precision (inverse of variance) \( s_i \). The information \( s_i \) comes at a cost \( c(s_i) \). We assume that \( c(\cdot) \) is a continuous, twice-differentiable function with \( c' > 0 \) and \( c'' \geq 0 \). The functional form of \( c(\cdot) \) implies that a more precise signal comes at a higher cost, and also that the marginal cost of a signal is increasing with its precision.

We assume that at \( t = 0 \) each agent \( i \) of the total \( N \) investors has 1 unit of the bond and \( x_i \) units of the risky stock, where \( x_i \) is drawn from a normal distribution with mean \( x_0 \) and variance \( N \times V \). The number of investors \( N \) is assumed to be large. The \( \tilde{x}_i \) are independent across investors, and also independent of the signals \( \tilde{y}_i \). In what follows, we will need the distribution of the per-capita supply of the risky asset, denoted by \( \tilde{X} \). It is given by
\[ \tilde{X} = \frac{\sum_{i=1}^{T} \tilde{x}_i}{N}, \]
Being a sum of normally distributed variables, it follows that \( \tilde{X} \) is normally distributed with mean \( x_0 \) and variance \( V \).

All investors have CARA utility, so for investor \( i \) we have
\[ U_i(w) = -\exp (-w/r_i). \]

There are two features that distinguish our framework from that of (Verrecchia 1982). First, we incorporate the wealth effect by setting the risk tolerance of investor \( i \) to be a linear function of her time-0 endowment \( x_i \):
\[ r_i = x_i/a, \ a > 0. \] (4)
This relationship implies that agents’s relative risk aversion parameter \( a \) remains constant, while their absolute risk aversion varies with wealth levels, as seen in CRRA utility.

\[ ^{11}\text{Distributing the bond holdings evenly makes the comparison of wealth levels easier, since stock prices are not yet defined at } t = 0 \text{ due to no trade occurring. Assuming symmetric bond holdings allows comparison by taking only number of stocks held rather than portfolio value.} \]

\[ ^{12}\text{The assumption that } N \text{ is large is needed to ensure that an individual investor does not affect the equilibrium prices and hence acts as a price-taker. This allows solving for the price in closed-form.} \]
Second, we assume that the average aggregate supply of the risky stock is positive. In the subsequent analysis we show that this assumption has important implications for the risk premium in the economy, and also for the relation between the initial wealth and fraction of wealth allocated to stocks.

3.2 Rational Expectations Equilibrium

In this section, we characterize the competitive equilibrium price and portfolio choice at \( t = 1 \), after all investors have observed the signals purchased previously. The formation of the equilibrium takes into account that the realized market price reflects the beliefs held and, conversely, the beliefs reflect the information portrayed by the price. This definition of the rational expectations equilibrium is standard and so we omit the details. The following proposition characterizes the equilibrium in our setting.

**Proposition 3.1.** The equilibrium price \( \tilde{P} \) converges in probability to

\[
\tilde{P} \to \alpha + \beta \tilde{u} - \gamma \tilde{X},
\]

where

\[
\begin{align*}
\alpha &= \frac{E[r]Vh_0\mu_0 + x_0E[r]E[rs]}{E[r]Vh_0 + E[rs]V + E[r](E[rs])^2}, \\
\beta &= \frac{E[rs]V + E[r](E[rs])^2}{E[r]Vh_0 + E[rs]V + E[r](E[rs])^2}, \\
\gamma &= \frac{V + E[r]E[rs]}{E[r]Vh_0 + E[rs]V + E[r](E[rs])^2}.
\end{align*}
\]

Several features of the competitive equilibrium are worth commenting on. First, expression \((5)\) is a probabilistic limit to which the equilibrium price converges when the number of investors \( N \) is very large. To understand what it means, we briefly outline the intuition, which hinges on the individuals’ market impact. (Hellwig 1980) refers to the model when \( N \) is finite as being “a bit schizophrenic”. Agents know that their actions affect the market price, but at the same time they behave as price takers. Letting \( N \) go to infinity overcomes this problem. Each investor is

---

13The interested reader can find them in (Verrecchia 1982) or (Hellwig 1980).
now truly atomistic and so the concept of competitive equilibrium becomes internally consistent.
The price in (5) is the limiting price for a sequence of competitive economies when $N \to \infty$.

Second, note that while no investor knows the realization of $\tilde{u}$, it enters the expression for
the market price. This stems from the fact that the noise of individual agents’ signals cancels
out in aggregation across all investors, again due to the ’large’ number of investors. Although
the market price explicitly depends on $\tilde{u}$, investors are not able to extract this information
because of the noise introduced through $\tilde{X}$.

Third, the equilibrium price depends on the realizations of two random variables: the return
on the risky asset $\tilde{u}$ and the per-capita supply of the risky asset $\tilde{X}$. The relative importance of
each of these two components is determined by the relation between $\beta$ and $\gamma$, where $\beta/\gamma = E[rs]$. Intuitively, $E[rs]$ is a measure of the informativeness of the market price. Indeed, it is obtained
by integrating over the individual precision choices weighted by the corresponding risk tolerance.
Such weighting is necessary because more risk tolerant investors hold more of the stock and thus
have a higher impact on the price informativeness. The intuition behind (5) is as follows. When
on average the investors are well-informed about the risky stock, they have a more precise
knowledge of $\tilde{u}$ and so the weight of this term in (5) is higher than when investors have less
information.

Now we are able to characterize the optimal portfolio choice by investors. After observing
the market price and her own signal at $t = 1$, each investor updates her beliefs about the mean
and the variance of the stock payoff. To compute the posterior values, we first need to know
the prior beliefs. They are given in the following Lemma.

\textbf{Lemma 3.1.} The vector $(\tilde{u}, \tilde{y}_i, P)$ of mean payoff, signal and $t = 1$ stock price has, as of $t = 0$, 
a jointly normal distribution with mean $(\mu_0, \mu_0, P_0)$, where

$$P_0 = \mu_0 - \frac{Vx_0}{E[\sigma]V + E[r]V + E[rs](E[rs])^2},$$

(8)
and variance-covariance matrix

\[
\begin{bmatrix}
\sigma_0^2 & \sigma_0^2 & \beta \sigma_0^2 \\
\sigma_0^2 & \sigma_0^2 + s_i^{-1} & \beta \sigma_0^2 \\
\beta \sigma_0^2 & \beta \sigma_0^2 & \beta^2 \sigma_0^2 + \gamma^2 V
\end{bmatrix}.
\]

The next proposition characterizes the optimal portfolio choice. We are able to solve for the pertinent quantities in closed form because \( \tilde{u} \), \( \tilde{y}_i \), and \( \tilde{P} \) are jointly normally distributed. This allows us to use well-known results on Bayesian updating under normality (see, e.g., (Gelman, Carlin, Stern & Rubin 2004)) to find the updated mean \( \mu_i \) and variance \( \sigma_i^2 \) of \( \tilde{u} \), conditional on investor \( i \) observing her signal \( \tilde{y}_i = y_i \) as well as the equilibrium price \( \tilde{P} = P_1 \).

**Proposition 3.2.** After observing her signal \( \tilde{y}_i = y_i \) and the market price \( \tilde{P} = P_1 \), investor \( i \)'s optimal number of stocks held, denoted by \( D_i \), is given by\(^{14}\)

\[
D_i = r_i \frac{\mu_i - P_1}{\sigma_i^2}
\]

where \( \mu_i \) and \( \sigma_i^2 \) are the posterior mean and variance of \( \tilde{u} \) given by

\[
\mu_i = \mu_0 + \frac{s_i \gamma^2 V (y_i - \mu_0) + \beta (P_1 - P_0)}{(h_0 + s_i) \gamma^2 V + \beta^2},
\]

\[
\sigma_i^2 = \frac{1}{h_0 + s_i + (\beta^2 / \gamma^2 V)}.
\]

In this Section we have completely characterized the portfolio choice for a given level of precision \( s_i \). We now turn to the analysis of how the investors choose the amount of information they are willing to buy.

### 3.3 Information Acquisition

In this section, we analyze the information acquisition problem that investors face. At time \( t = 0 \), investors choose the precision of their signal, \( s_i \), by maximizing their expected utility from terminal wealth, which depends on the portfolio choice to be made at \( t = 1 \). The investors

---

\(^{14}\)We do not present explicitly the demand for the bond as we will not need it in the subsequent analysis. It is easily obtained from the budget constraint.
are myopic in the sense that when deciding how much information to buy at \( t = 0 \), conditional on future portfolio choice, they should actually anticipate that their wealth, and thus their absolute risk aversion \( r_i \), will have changed by \( t = 1 \) due to realized stock prices, yet they don’t. They simply use their ”endowment” absolute risk aversion, i.e. \( r_i = w_{0i}/a \), as the relevant parameter. Of course, when period 1 arrives, the optimal portfolio will be based on the time-1 level of wealth through \( r_i = w_{1i}/a \).

Having assumed that investors are myopic, Verrecchia’s result goes through without fundamental changes. The only change we need to make is to substitute the risk tolerance by \( w_{0i}/a \).

**Proposition 3.3.** There exists a unique competitive information acquisition equilibrium. Investor \( i \)’s optimal choice of the precision \( s_i \) is given by \( \max[0, \hat{s}_i] \), where \( \hat{s}_i \) is implicitly given by

\[
\frac{2ac'(\hat{s}_i)}{w_{0i}} \left[ \hat{s}_i + h_0 + \frac{(E[rs])^2}{V} \right] = 1.
\]

(Verrecchia 1982) shows that the level of precision chosen is a nondecreasing function of risk tolerance. The reason why the function is nondecreasing, as opposed to increasing, is that for risk tolerance below a certain threshold, investors do not buy any information at all. Hence, a marginal increase of the risk tolerance leaves the optimal precision at the same level – zero. When an investor chooses to buy a positive amount of information, increasing her absolute risk tolerance always leads to purchasing strictly more information. In our case, given the link between risk tolerance and wealth, we have that: a) there exists a wealth threshold such that only agents whose initial wealth lies above the threshold acquire information about the stock, b) when the initial wealth is above the threshold, increasing initial wealth leads to more information being purchased. The qualitatively same result is obtained in (Peress 2004) numerically. Conclusions regarding the impact of other exogenous parameters on the amount of acquired information are also qualitatively identical in the two models. Namely, the level of precision is a non-increasing function of a) price informativeness, b) marginal cost of information, c) relative risk aversion \( a \). While the results of the two models on information acquisition are

---

\( ^{15} \)See Lemma 2, p. 1421 in (Verrecchia 1982).

\( ^{16} \)See Corollary 1, p. 1423
the same, our analysis is a straightforward, tractable extension of (17). This confirms that characteristics lost by moving from CRRA to CARA do not play a central role for questions of information acquisition. Even for the "extended CARA" setting, the attractive simplicity of CARA for information acquisition holds and proofs remain simple.

From Proposition 3.3 it follows that an individual decision on how much information to acquire depends on the aggregate amount of information purchased by all investors, through the term $E[rs]$, a measure of price informativeness. Hence, one needs to prove that the equilibrium exists. Verrecchia proves the existence but not uniqueness. As shown in the proof to the following proposition, here we complement the analysis by showing uniqueness.

**Proposition 3.4.** The information acquisition equilibrium is unique.

proof: see appendix.

### 3.4 Wealth and Portfolio Shares

We now turn to the main economic question of this Section – understanding how initial endowment affects $\theta_i$, the share of wealth invested into the risky security. Specifically, we are interested in how the unconditional share invested, i.e. the expected value of $\theta_i$ as of $t=0$, depends on initial wealth. The next Proposition presents the main result of Section 3.3.

**Proposition 3.5.** The effect of the purchased signal's precision on the share of wealth expected to be allocated to stocks is given by

$$
\frac{dE_0[\theta_i(s_i)\]}{ds_i} = \frac{P_0(\mu_0 - P_0) + \beta (\sigma_0^2 - \beta \sigma_0^2) - \gamma^2 V}{a}.
$$

(13)

The sign of $dE[\theta_i(s_i)]/ds_i$ is ambiguous.

The initial wealth affects $E[\theta_i]$ only through $s_i$, since the adjustment to the CARA utility function leads the optimally invested wealth share being constant. In the region of low wealth, in which investors do not buy any information, marginally increasing the initial wealth has no

---

17 Strictly speaking, the proof for the existence of the equilibrium at $t = 1$ in Verrecchia is not directly applicable in our framework because he assumes that the investors’ risk tolerance belong to a compact set, while we assume that initial wealth (and hence risk tolerance) has infinite support due to normality. To get around this problem, we restrict $r_i$ to belong to $[r_{\text{min}}, K]$, where $r_{\text{min}} > 0$ is small and $K >> r_{\text{min}}$. So if initial wealth of investor $i$ is lower than $r_{\text{min}}$ (higher than $K$) we assume that $r_i = r_{\text{min}} (r_i = K)$. This is clearly without loss of generality.
effect on the signal and, hence, on the wealth share allocated to stocks. On the other hand, when 
the initial wealth is high enough so as to make investors buy a positive amount of information, 
the sign of $dE[\theta_i(s_i)]/ds_i$ is the same as the sign of $dE[\theta_i(s_i)]/dw_{0i}$.\footnote{In other 
words, since $dE[\theta_i(s_i)]/dw_{0i} = dE[\theta_i(s_i)]/ds_i \times ds_i/dw_{0i}$, and $ds_i/dw_{0i}$ is always 
non-negative, we will always expect $\theta_i$, the share invested into the risky asset, to increase with 
wealth iff it increases with signal’s precision.} From Proposition 3.5 we see that the effect of higher $s_i$ 
on the expected fraction invested into the stock is ambiguous. Indeed, while the first two terms therein are positive (because $\mu_0 > P_0$ and $\beta < 1$), if the term $\gamma^2V$ is large enough, the total effect can be negative.

This result is closely linked to our assumption on positive net supply of the asset. In contrast, 
Verrecchia assumes a zero net supply $x_0 = 0$, thereby eliminating any expected equilibrium risk 
premium. Indeed, as follows from (8), $x_0 = 0$ implies $P_0 = \mu_0$, making the vector of Proposition 
3.1 take on the form $(\mu_0, \mu_0, \mu_0)$.\footnote{The intuition for a risk premium of zero is as follows. A random variable $\tilde{X}$ is symmetric around 0. So for 
each positive realization of $\tilde{X} = \hat{X}$ there is a corresponding negative realization $-\hat{X}$ occurring with the same 
probability. The latter is essentially the mirror case of the former, so on average the risk premium is zero.} The next Corollary highlights the importance of the positive risk premium for the ambiguity result.

**Corollary 3.1.** If the average risk premium is zero, then the average wealth share invested into 
the risky stock decreases with precision, i.e.

$$
\frac{dE[\theta_i(s_i)]}{ds_i} < 0.
$$

Hence, the wealth share is a non-increasing function of the initial wealth.

The Corollary reveals a surprising result that in the original Verrecchia’s setting the effect 
of initial endowment on the wealth share is always negative.

To understand the intuition behind Proposition 3.5 and Corollary 3.1, let us decompose the 
expectation of $\theta_i$ as follows:

$$
E[\theta_i] = E\left[ \frac{P_1(\mu_i - P_1)}{a\sigma_i^2} \right] = E[P_1]E\left[ \frac{\mu_i - P_1}{a\sigma_i^2} \right] + \text{cov} \left[ P_1, \frac{\mu_i - P_1}{a\sigma_i^2} \right].
$$

(14)

From (10), the covariance between $\mu_i$ and $P_1$ decreases as $s_i$ increases. As investor $i$ has a 
more precise private signal, he pays less attention to the information conveyed by the market
price and trusts his own signal more. Hence, investor $i$ updates his estimate $\mu_i$ much less in reaction to changes in $P_1$ when $s_i$ is high. In the limit, when $s_i$ tends to infinity, the updated expectation of payoff $\tilde{u}$, $\mu_i$, depends only on the private signal. As a result, the covariance term in (14) decreases with $s_i$. Whether the first term in (14) increases or remains the same depends on the expected risk premium and, thus, on the average supply of the risky asset. If the average risk premium is zero, the risk averse investor will hold a zero position in the stock on average, regardless of his signal’s precision. Hence, the first term disappears and only the negative correlation term is left – hence, the result of Corollary 3.1. When the average supply of the stock is positive, the equilibrium risk premium is increased. This makes the stock attractive for risk averse investors and they hold a positive amount of it on average. Now having more precise information about the stock’s payoff increases the wealth share invested into the stock.

So the first term in (14) becomes positive. The combined effect of the two terms is ambiguous – as stated in Proposition 3.5.

This result is at odds with the conclusion of (Peress 2004) who demonstrates an unambiguous effect – higher chosen precision always leads to investors expecting to invest a higher share of wealth into the risky stock. The difference stems from the fact that the solution based on CRRA utility is obtained using a log-linear approximation. As the logarithmic transformation is non-affine, it distorts the correlations between random variables which leads to qualitatively different results. As follows from the above intuition, the covariance between the random variables $\mu_i$ and $\tilde{P}$ plays an important role here. If we are to take the logarithm of $\theta_i$ – ignoring for the sake of the argument that $\theta_i$ can be negative and so this operation may not be well defined – then we would get $\ln(P_1) + \ln(\mu_i - P_1) - \ln(\sigma_i^2)$. So we move from the multiplicative relation to the additive one, implying that the (negative) correlation effect disappears.

A natural way to resolve the indeterminacy described in Proposition 3.5 is calibrating the model’s parameters from the data. However, in this particular case the calibration could be quite a tricky exercise. For example, we need to know the cost function $c(s)$ which relates the money spent to the precision of the signal. While the concept of the signal defined in value by its precision is theoretically appealing, it is difficult to relate it to some real-world observable variable.

This result regarding the implications of log-linearization is certainly not new. The argument
typically made to justify this transformation is that it affects mainly the second moments so
the resulting error is likely to be small. However, some papers find that the approximation
errors due to the log-linearization can in fact be significant. For example, (Kim & Kim 1999)
demonstrate that log-linearization can lead to spurious welfare results for reasonable values
of model parameters. All in all, we believe that assuming away the impact of correlations
particularly in an informational setting – which by construction relies mainly on the covariances
between fundamentals, signals and resulting equilibrium values – is an inconsistent route to take.

4 Information and Portfolio Choice: a Multiple-Priors Approach

Another prominent strand of literature that relies on CARA specification is that of portfo-
lio choice among ambiguity-averse (or uncertainty-averse) investors. The setup of (Gilboa
& Schmeidler 1989) is one of the most commonly used models of this type of utility function, e.g.
in (Kogan & Wang 2002), (Garlappi et al. 2007), and (Cao et al. 2005). Ambiguity aversion
goes beyond parameter uncertainty in that it purports that investors are not only unsure of
the true parameters (often the mean) of the return distribution, but also of what the correct
distribution is. In the (Gilboa & Schmeidler 1989) setup, this essentially leads to kinked indif-
ference curves. But similarly to Grossman & Stiglitz type models, the combination of negative
exponential utility and normal returns are crucial for tractability. In this section we study how
incorporating the wealth effect changes the results of (Cao et al. 2005).

4.1 Economic Setting

Again, there are two assets in the economy – a risky stock and a risk-free bond. The risk
free rate is normalized to zero. The stock’s payoff $\tilde{u}$ is normally distributed with mean $\mu$ and
variance $\sigma$. While investors know the exact value of $\sigma$, we assume that they are uncertain
about the true value of $\mu$ and base their decisions on what they perceive as the “worst-case
scenario” ((Gilboa & Schmeidler 1989)). We assume that investors believe that the true mean
of $\tilde{u}$ belongs to the set $[\mu - \phi_i, \mu + \phi_i]$, where $\phi_i$ defines the level of uncertainty for investor $i$. $\phi_i$
is uniformly distributed on the interval $[\bar{\phi} - \delta, \bar{\phi} + \delta]$, so $\delta$ is a measure of uncertainty dispersion
across investors.
Cao et al. describe several possible reasons why heterogeneity in levels of uncertainty may arise, e.g. different proprietary models used to analyze data. Our interpretation of the heterogeneity is related to information acquisition. To keep things simple and focus on the portfolio choice of investors, we do not provide a formal analysis of the information acquisition in this setting. Instead, we rely on the standard reasoning on how wealth affects the incentives to buy information. As discussed previously in the paper, wealthier investors will optimally acquire more information. Buying more information leads to a greater reduction in the level of uncertainty, in other words it will result in a narrower interval around the true mean $\mu$.  

We now introduce wealth heterogeneity and assume that investor $i \in [0, 1]$ is endowed with $x_i$ units of risky stock. We assume the following relationship between the initial wealth and the level of uncertainty:

$$\phi_i = \frac{1}{x_i}.$$  (15)

This relationship expresses the idea that if investor $i$ has little wealth, she will spend little on reducing her uncertainty about the stock market, so remains with a higher value of $\phi_i$. While there are other possible functional forms for the link between $x_i$ and $\phi_i$ (e.g., a decreasing linear function), (15) is reasonable in a sense that, however high his wealth is, an investor is not able to learn the true $\mu$ precisely. This would not be the case if $x_i(\phi_i)$ were a linear or concave function.

Another realistic feature of the relation (15) is the implied distribution of wealth. As we assumed that $\phi_i$ is uniformly distributed, from (15) it follows that wealth in our model has a power distribution $\propto w^{-1}$ which is broadly consistent with the empirical evidence. A feature of this distribution is that fewer people are associated to increasing levels of wealth. This property is sometimes referred to as “80-20 rule” meaning that 20% of the population owns 80% of the wealth.

---

20 A theoretical model of how information is transformed into a narrower ambiguity interval can be found in (Epstein & Schneider 2006). They show that learning leads to the ambiguity interval shrinking over time. Learning over a larger time interval means observing more data and is thus equivalent to having more information in our context.

21 Based on the insights developed in the analysis of the Bayesian setup, we believe that a rigorous model of the information acquisition process can reveal that the relationship between $x_i$ and $\phi_i$ is only weakly monotonic. That is, in the region of the low wealth agents will not purchase information at all and hence $\phi_i$ will be the same for a set of investors with different (low) wealth. Introducing this feature will not change our main conclusions.

22 This inversely proportional distribution is a special case of a more general Pareto distribution often used to describe the distribution of wealth in various countries. See, e.g. (Persky 1992).
The average supply of the risky stock, denoted by $\bar{x}$, is determined as follows in this setup.

$$\bar{x} = \int_{\bar{\phi} - \delta}^{\bar{\phi} + \delta} \frac{1}{\phi} \frac{1}{2\delta} d\phi_i = \frac{1}{2\delta} \ln \frac{\bar{\phi} + \delta}{\bar{\phi} - \delta}$$  \hspace{1cm} (16)

As before, we introduce the wealth effect in our CARA-normal setting by relating the investor’s absolute risk tolerance to her initial endowment:

$$r_i = x_i/a.$$  \hspace{1cm} (17)

### 4.2 Portfolio Choice

As before, we denote by $P$ the stock price and by $D_i$ investor $i$’s demand for the stock. The optimal portfolio is standard and so is given without derivation.\footnote{For the details, see Section 3.1 in Cao et al. (2005).} It is given by:

$$D_i = \begin{cases} \frac{r_i}{\sigma^2} (\mu - \phi_i - P) & \text{if } \mu - P > \phi_i, \\ 0 & \text{if } -\phi_i \leq \mu - P \leq \phi_i, \\ \frac{r_i}{\sigma^2} (\mu + \phi_i - P) & \text{if } \mu - P < -\phi_i. \end{cases}$$  \hspace{1cm} (18)

The fraction of wealth invested into the stock, $\theta_i$, is given by

$$\theta_i = \frac{PD_i}{w_{0i}}.$$  

Because $w_{0i} = P x_i$ we have that

$$\theta_i = D_i/x_i.$$  \hspace{1cm} (19)

Finally, combining (17), (18), and (19) we have

$$\theta_i = \begin{cases} \frac{1}{\sigma^2} (\mu - \phi_i - P) & \text{if } \mu - P > \phi_i, \\ 0 & \text{if } -\phi_i \leq \mu - P \leq \phi_i, \\ \frac{1}{\sigma^2} (\mu + \phi_i - P) & \text{if } \mu - P < -\phi_i. \end{cases}$$  \hspace{1cm} (20)
4.3 Equilibrium with Full Participation

First, we analyze the case of full market participation. This case corresponds to the situation when the market price is sufficiently low so that even investors with the highest uncertainty (the highest $\phi_i$) participate in the stock market. From (15), (17), and (18) the demand of investor $i$ in this case is

$$D_i = \frac{1}{a\phi_i \sigma^2} (\mu - \phi_i - P)$$

The next proposition characterizes the equilibrium price.

**Proposition 4.1.** The equilibrium price with full participation, $P$, is given by

$$\mu - P = a\sigma^2 + \frac{1}{\bar{x}}$$  \hspace{1cm} (21)

It is of interest to compare (21) with the corresponding equation derived by Cao et al. In their case, the pricing equation (using our notation) is

$$\mu - P = a\sigma^2 \bar{x} + \bar{\phi}.$$  \hspace{1cm} (22)

The first terms on the right-hand side of (21) and (22) represent the risk premium which is proportional to the relative risk aversion $a$ and the stock’s variance. However, in (22), unlike our expression (21), the risk premium depends on the average supply of the risky stock. Due to lack of a wealth effect, the aggregate demand is independent of the stock’s supply. So as $\bar{x}$ increases, the risky stock has to become more attractive for the market to clear. This is achieved through increasing the equity premium. In the case studied here, the increase in supply is matched by the corresponding increase in absolute risk tolerance and so investors are happy to hold a larger position in the stock at the same price.

The second term in Cao et al. represents the premium for ambiguity and it depends on the average uncertainty in the economy $\bar{\phi}$, while the uncertainty dispersion $\delta$ does not affect the equilibrium price. In our model, which takes into account the interaction of wealth and information, the premium for uncertainty is inversely proportional to the average supply of the risky asset which is a proxy for the average initial wealth. The intuition is straightforward –
when the wealth of an average investor increases, she buys more information about the stock which reduces the average uncertainty in the economy. As a result, the uncertainty premium component of the price decreases.

Notice that from (4.3) it follows that $\bar{x}$ is related to the uncertainty dispersion $\delta$, and hence the finding that only the average uncertainty matters for the equilibrium risk premium does not survive in a setting where the wealth effect is taken into account. Differentiating $\bar{x}$ with respect to $\delta$ yields that $\bar{x}$ increases with $\delta$.

[?!?!?really? How can this be true in general from eq ?!?!]

Hence, higher uncertainty dispersion leads to a lower ambiguity premium. Looking at the effect of average uncertainty $\bar{\phi}$ on the ambiguity premium, we get the same result as Cao et al. – the uncertainty premium increases with $\bar{\phi}$.

In order for all potential investors to participate in the stock market, the model parameters should satisfy certain conditions. When the investor with the highest uncertainty holds a long position in stock, the other – less uncertain – investors will also participate in the stock market. Hence, when

$$\mu - (\bar{\phi} + \delta) - P > 0,$$

we have full participation. Denote the wealth of the poorest investor by $x_{\text{min}}$. From (15) it follows that $x_{\text{min}} = 1/(\bar{\phi} + \delta)$. Therefore full participation implies

$$\frac{1}{x_{\text{min}}} - \frac{1}{\bar{x}} < a\sigma^2,$$

i.e. it is likely to occur when the dispersion of the initial wealth is relatively low.

4.4 Equilibrium with Limited Participation

When wealth dispersion is high, some investors may choose not to hold the risky stock. This is a notable distinction from the Bayesian setting, presented in the first part of the paper, since in that case non-participation will occur only in cases of measure zero. Denote by $\phi^*$ the threshold level of uncertainty so that investors with a higher level of uncertainty, $\phi_i > \phi^*$, do not participate in the stock market, while the rest do. As follows from (18), the threshold $\phi^*$ is
found by looking at the marginal investor choosing to invest:

$$\mu - \phi^* - P = 0.$$  \hspace{1cm} (25)

The next proposition characterizes the equilibrium $P$ and $\phi^*$.

**Proposition 4.2.** In the equilibrium with limited market participation, the threshold value of uncertainty $\phi^*$ is implicitly given by

$$\ln \frac{\phi + \delta}{\phi - \delta} = \frac{\phi^*}{a\sigma^2} \ln \frac{\phi^*}{\phi - \delta} - \frac{1}{a\sigma^2} (\phi^* - \phi + \delta).$$  \hspace{1cm} (26)

The equilibrium price is given by

$$P = \mu - \phi^*.$$  \hspace{1cm} (27)

[WHY? How does eq. 26 arise?!! proof not clear enough...]

We now turn to analyzing the model’s predictions and relating them to the empirical evidence.

### 4.5 Wealth Share, Market Participation and Risk Premium

Given the fact that even in developed countries a large fraction of households do not participate in the stock market, we focus mainly on which observed investment patterns can be explained by this limited participation equilibrium. Below we look at three features: 1) stock market participation and its relation to wealth, 2) wealth share invested into the risky asset, 3) equity premium.

**Stock Market Participation**

The investors with high uncertainty are more likely not to participate in the stock market. Given the link between initial wealth and uncertainty, it is the poor households that stay away from the stock market in our model, which is consistent with the empirical evidence.

It is not unusual to observe non-participation even among the households whose wealth exceeds $100,000, as shown by (Mankiw & Zeldes 1991). From our model it follows that if an investor with some wealth $A$ participates in the stock market, then an investor with wealth $B$,
such that $B > A$, will also participate. While this seems to be at odds with the findings of Mankiw and Zeldes, this model could easily be extended to account for this fact. Remember that we assumed that investors’ uncertainty intervals are symmetric around the true $\mu$. This means that the worst-case mean return of a rich investor, whose interval is narrow, is always higher than the worst-case mean return of a poor investor, whose interval is wide. In this sense, a rich investor in our model always always seems more optimistic than a poor one $^{24}$ As a result, we cannot have a situation with a poor investor investing and a rich investor not investing.

However, in the style of Bayesian updating, different investors may have different prior means around which their ambiguity intervals are centered. Under this assumption it is possible to have a situation of a wealthy investor, while having less ambiguity, being more pessimistic in the “worst-case scenario” sense than a poor one. Given this, a wealthier household may optimally have zero holdings in the stock market, while a poorer one participates in the stock market. We believe that such an extension would not change our main qualitative results $^{25}$. Indeed, we will still have that the minimal value (left boundary) for a narrower interval is higher on average than that for a wider interval $^{26}$. So wealthier households will on average still be more optimistic about the risky stock’s payoff – consistent with our assumptions.

To investigate the predictions of our model regarding stock market participation, we look at the proportion of participating investors, denoted by $\pi$, where

$$\pi = \frac{\phi^* - (\hat{\phi} - \delta)}{2\delta}.$$  \hspace{1cm} (28)

As a result of economic growth, households’ wealth tends to increase over time. To look at this within an essentially static model, it is most natural to analyze the effect of higher initial endowments on the stock market participation. Suppose the initial stock endowment of all investors increases by a factor of $k$, so that the new endowment of investor $i$ equals $k \times x_i$.

In terms of the ensuing uncertainty, this affects both the average level of uncertainty, which

$^{24}$This is the case for long positions being held, i.e. with positive net supply assets.

$^{25}$However, we do not deal with it in detail, due to the well-known axiomatic problems of combining learning behavior and ambiguity aversion [SOME CITATIONS??]

$^{26}$To understand why it is so, consider the following simple example. Suppose that after purchasing the information the resulting uncertainty interval has a width $Y$ and is equally likely to have any position around the true $\mu$. As is easy to see, the left boundary of the interval is uniformly distributed between $\mu - Y$ and $\mu$. Hence, the average worst case scenario is given by $\mu - Y/2$. The wider the interval, i.e. the higher $Y$, the lower is the average worst-case value.
becomes $\bar{\phi}/k$, and the uncertainty dispersion, which becomes $\delta/k$.

**Proposition 4.3.** For the equilibrium market participation $\pi$, we have that

$$\frac{d\pi}{dk} > 0$$

As follows from Proposition 4.3, the proportional increase of the investors’ initial endowments leads to higher stock market participation. Indeed, from (24) it follows that a proportional increase of both $x_{min}$ and $\bar{x}$ decreases the left-hand side of this inequality, thus moving the economy towards the full-participation scenario.

A recent comprehensive study of the household stockholding in Europe by (Guiso et al. 2003) documents that stock market participation increases over time. Proposition 4.3 provides an explanation for this finding.

An alternative but related explanation for limited participation also described in (Guiso et al. 2003), involves entry costs. The increase in participation is explained by the fact that these costs have been decreasing over time as a result of the increasing competition among financial institutions. Another cost-based explanation concerns transaction costs. For example, it used to be expensive to have a well-diversified portfolio of stocks since transaction costs were incurred on each individual stock. Now there are many mutual funds that allow any investor to own a certain index at a small cost.

However, cost-based explanations cannot account for some features of the data. For example, many households with high wealth, for whom entry costs are a very small fraction of the assets, do not participate in the stock market. Models with uncertainty-averse investors, such as the one analyzed in this paper, are able to explain such phenomenon. However, for our explanation to work it is essential that investors are “sufficiently” heterogeneous in terms of their level of ambiguity. Otherwise, the full participation case is likely to occur.

(Welch 2000) reviews several papers on the estimation of the equity premium and concludes: “Unfortunately, there is neither a uniformly accepted precise definition nor agreement on how the equity premium should be computed and applied.” Given that even academics and professionals studying the stock market cannot agree on how to estimate the equity premium, it is natural to expect a great deal of heterogeneity across households, including wealthy ones,
regarding the precision of their estimates of $\mu$.

**Wealth Share**

Our model implies that wealth share invested into the stock market increases with wealth, as ambiguity is reduced with further purchases of information. Unlike the setting presented in the first part of the paper, the effect is now unambiguous.

One of the findings of (Guiso et al. 2003) is that initial wealth has a positive but small effect on the asset share invested in the stock market – for those investors who do participate. They interpret this evidence as supporting the relevance of participation costs. The reasoning is that, while wealth is important for deciding whether to participate in the stock market or not, once investors have incurred these costs there is not much difference in their stockholdings.

[?!?! This does not sound reasonable. They are not the same, not CRRA - the wealthier, the more.??!]

Our model provides an alternative explanation for this finding. Remember that the relationship between the initial wealth and the level of uncertainty is decreasing and convex. So in the region of high wealth changing the initial endowment has a small effect on the level of uncertainty and, hence, on the portfolio choice.

**Equity Premium**

Various studies investigate the relationship between limited participation and equity premium. Some papers assume, without modelling the underlying mechanisms, that some investors do not participate in the stock market, for example (Basak & Cuoco 1998), (Mankiw & Zeldes 1991), and (Brav, Constantinides & Geczy 2002). These studies suggest that limited participation increases the equilibrium equity premium compared to the full-participation case and hence can help resolve the equity premium puzzle as described by (Mehra & Prescott 1985).

Surprisingly, in analyzing their model, in which the decision whether to participate or not is endogenous, (Cao et al. 2005) et al. show that the opposite is true. They show that increasing the uncertainty in the economy [dispersion?! Their premium is indep of dispersion, no?!?!] decreases both the participation and the equity premium. This implies that limited participation in fact makes the equity premium puzzle even worse.
The insights from our model can reconcile these findings. While Cao et al. look primarily at ambiguity aversion and the endogeneity of the participation decision, the underlying utility specifications in the two approaches are also different: Basak and Cuoco use CRRA, whereas Cao et al. use CARA. Our model allows us to combine the nice features of the two settings: we have both endogenous participation and wealth effect.

The equilibrium in our model depends on several parameters, and changing each parameter is likely to affect both the participation rate and the risk premium. Cao et al. choose to vary the uncertainty dispersion $\delta$. However, in our model $\delta$ is not an exogenous parameter but rather is determined by the value of the initial endowment, via the information acquisition. For this reason, when looking at comparative statics, we alter the investors’ endowments.

As before, we consider a proportional increase $k > 1$ of the investors’ endowments. As we have already shown, this leads to an increase in stock market participation due to more information being purchased. In the next proposition we look at the effect on the equity premium.

**Proposition 4.4.** Suppose the initial stock endowment of all investors is multiplied by a factor of $k > 1$. In equilibrium, the equity premium will fall, as

$$\frac{d(\mu - P)}{dk} < 0.$$ 

As follows from Proposition 4.4 we reach the same conclusion as Basak and Cuoco – decreasing stock market participation leads to a higher equity premium. Our results are also consistent with the empirical evidence that the equity premium has been steadily declining over several decades, (see (Blanchard, Shiller & Siegel 1993), (Fama & French 2002), and (Jagannathan, McGrattan & Scherbina 2000)) while the stock market participation has been increasing (see (Bertaut & Starr-McCluer 2000) and (Mankiw & Zeldes 1991)).

5 Conclusion

In this paper we propose a simple method to account for the wealth effect in models with CARA utility. The idea is to explicitly link an investor’s absolute risk aversion to her wealth.
We then apply this approach to investigate two models of portfolio choice in the presence of costly information.

First, we incorporate the wealth effect into the framework of (Verrecchia 1982) and examine the ability of the resulting model to explain why wealthier households invest a larger fraction of their wealth into risky assets. Unlike (Peress 2004), we find that in a learning environment, the effect of the initial wealth on portfolio shares is ambiguous. The difference between the two seemingly similar models stems from the fact that log-linearization is required for the CRRA model, while our approach leads to an analytic solution. Log-linearization, being a non-affine transformation, distorts the correlation between pertinent random variables. The models are not calibrated in this paper, as some of the parameters needed for calibration, e.g. the level of price informativeness, are difficult to estimate from the data. Thus, whether incorporating the wealth effect into (Verrecchia 1982) can indeed explain the fact that wealthier individuals seem to invest a larger share of their wealth into risky securities, is still an open question. On the methodological front, this paper highlights the attractive properties of our approach based on adjusting CARA utility functions, compared to changing the functional form of the utility to CRRA. Our main message is that incorporating the wealth effect into the existing models with CARA utility may not be as formidable task as previously thought.

Second, we incorporate the wealth effect into the (Cao et al. 2005) framework. The resulting model explains several salient features of households’ stockholding. We show that if learning – which is determined by wealth level – decreases the uncertainty faced by investors, the wealth share they invest into risky assets now unambiguously increases with wealth. In addition, the model predicts that wealthier households are more likely to participate in the stock market than poorer ones. Finally, the model provides an explanation for the fact that market participation increases over time, while the equity premium decreases.
A Appendix

Proof of Proposition 3.1

See Proposition 5.2 in Hellwig (1980), p. 492.

Q.E.D.

Proof of Lemma 3.1

First, we derive the means of the random variables. By assumption, \( E[\tilde{u}] = \mu_0 \). For the signal, we have \( E[\tilde{y}_i] = E[\tilde{u}] + E[\tilde{\epsilon}_i] = \mu_0 \). For the equilibrium price, we have

\[
E[\tilde{P}] = E[\alpha + \beta \tilde{u} - \gamma \tilde{X}] = \alpha + \beta \mu_0 - \gamma x_0 \\
= \frac{(E[r]V h_0 \mu_0 + x_0 E[r] E[rs]) + (E[rs]V \mu_0 + E[r] (E[rs])^2 \mu_0) - (V x_0 + E[r] E[rs] x_0)}{E[r] V h_0 + E[rs] V + E[r] (E[rs])^2} \\
= \frac{\mu_0 (E[r] V h_0 + E[rs] V + E[r] (E[rs])^2) - V x_0}{V x_0} \\
= \mu_0 - \frac{E[r] V h_0 + E[rs] V + E[r] (E[rs])^2}{V x_0}.
\]

Now we derive the variance-covariance matrix. By assumption,

\[
Var[\tilde{u}] = h_0^{-1}, \\
Var[\tilde{y}_i] = \text{Var}[\tilde{u}] + \text{Var}[\tilde{\epsilon}_i] = h_0^{-1} + s_i^{-1}. \\
\]

For the equilibrium price, we have

\[
\text{Var}[\tilde{P}] = \text{Var}[\beta \tilde{u}] + \text{Var}[\gamma \tilde{X}] = \beta^2 h_0^{-1} + \gamma^2 V.
\]

Finally,

\[
\text{Cov}[\tilde{u}, \tilde{y}_i] = \text{Cov}[\tilde{u}, \tilde{u} + \tilde{\epsilon}_i] = h_0^{-1}, \\
\text{Cov}[\tilde{u}, \tilde{P}] = \text{Cov}[\tilde{u}, \alpha + \beta \tilde{u} - \gamma \tilde{X}] = \beta h_0^{-1}, \\
\text{Cov}[\tilde{y}_i, \tilde{P}] = \text{Cov}[\tilde{u} + \tilde{\epsilon}_i, \alpha + \beta \tilde{u} - \gamma \tilde{X}] = \beta h_0^{-1}.
\]

Q.E.D.
Proof of Proposition 3.2.
We could present a full-blown proof here, but in principle the expressions are the same as in Verrecchia, p. 1420, with the only difference being the average equilibrium price: in his setting it equals $\mu_0$ and so the posterior mean has the term $\beta(P - \mu_0)$, while in our setting it equals $\mu_0 - Vx_0/(E[r]Vh_0 + E[rs]V + E[r](E[rs])^2)$ and so we have this expression instead of $\mu_0$.

Q.E.D.

Proof of Proposition 3.3.
See the proof of Lemma 2 in Verrecchia (1982), p. 1421.

Q.E.D.

Proof of Proposition 3.4.
First, notice that the uniqueness of the equilibrium is equivalent to the uniqueness of price informativeness $\Delta = (E[rs])^2/V$. In other words, it is not possible to have two different levels of precision, $s^1 \neq s^2$, $s^1 > 0$, $s^2 > 0$, such that both satisfy (12). This follows from the fact that $\frac{2c'(s_i)}{r_i} \left[ \hat{s}_i + h_0 + \frac{(E[rs])^2}{V} \right]$ strictly increases in $\hat{s}_i$ and so there is (at most) one positive value of precision such that (12) is satisfied.

Suppose there are two different equilibrium levels of price informativeness, $\Delta_1 > \Delta_2$. As follows from Corollary 2 in Verrecchia (p.1424), the amount of acquired information is a nonincreasing function of the price informativeness. That is, with the informativeness $\Delta_1$ some agents purchase the same amount of information and some agents purchase less information than with $\Delta_2$. We get the contradiction: higher $\Delta_1$ implies that agents have less precise signals, which means that $E[rs]$ decreases. But by definition $\Delta = (E[rs])^2/V$ and so $\Delta$ should also decrease. Hence, it is not possible to have $\Delta_1 > \Delta_2$.

Q.E.D.

Proof of Proposition 3.5.
Plugging (10) and (11) into (9), after some algebra, yields:

$$D_i = r_i \left( \mu_0 \left( h_0 + \frac{\beta^2}{\gamma^2 V} \right) + s(y_i - P) + P \left( \frac{\beta}{\gamma^2 V} - h_0 - \frac{\beta^2}{\gamma^2 V} \right) - P_0 \frac{\beta}{\gamma^2 V} \right).$$
Because the fraction of wealth invested into the risky stock is defined as \( \theta_i \equiv (PD_i)/w_i \) and because \( r_i = w_i/a \), we have

\[
\theta_i = \frac{P}{a} \left( \mu_0 \left( h_0 + \frac{\beta^2}{\gamma^2 V} \right) + s(y_i - P) + P \left( \frac{\beta}{\gamma^2 V} h_0 - \frac{\beta^2}{\gamma^2 V} \right) \right).
\]

We have

\[
\frac{dE[\theta_i(s_i)]}{ds_i} = E[d\theta_i(s_i)/ds_i] = (1/a)E[P(y_i - P)] = (1/a)(E[P]E[y_i] + cov[P,y_i] - (E[P])^2 - var[P]).
\]

Now we make use of the fact that \( P \) and \( y_i \) are jointly normal with known mean and covariance matrix derived in Lemma 3.1. This allows us to write the last expression as

\[
\frac{dE[\theta_i(s_i)]}{ds_i} = \frac{P_0\mu_0 + \beta\sigma_0^2 - P_0^2 - \beta^2\sigma_0^2 - \gamma^2 V}{a} = \frac{P_0(\mu_0 - P_0) + \beta(\sigma_0^2 - \beta^2\sigma_0^2) - \gamma^2 V}{a}. \tag{29}
\]

\[Q.E.D.\]

**Proof of Corollary 3.1.**

Substituting \( x_0 = 0 \) in \[8\] yields that \( \mu_0 - P_0 = 0 \). Hence, from \[29\] it follows that the sign of \( dE[\theta_i(s_i)]/ds_i \) is the same as the sign of

\[
\beta(\sigma_0^2 - \beta^2\sigma_0^2) - \gamma^2 V. \tag{30}
\]

First, let us look at \( \beta(\sigma_0^2 - \beta^2\sigma_0^2) \). Substituting \[6\] into this expression and combining the fractions using the common denominator \((E[r]Vh_0 + E[rs]V + E[r](E[rs])^2)^2\), we get:

\[
\frac{E[rs]V + E[r](E[rs])^2}{(E[r]Vh_0 + E[rs]V + E[r](E[rs])^2)^2} \frac{(E[r]Vh_0 + E[rs]V + E[r](E[rs])^2)^2\sigma_0^2 - (E[rs]V + E[r](E[rs])^2)^2\sigma_0^2}{(E[r]Vh_0 + E[rs]V + E[r](E[rs])^2)^2}
\]

\[
= \frac{\sigma_0^2(E[rs]V + E[r](E[rs])^2)[E[r]Vh_0 + E[rs]V + E[r](E[rs])^2 - E[rs]V + E[r](E[rs])^2]}{(E[r]Vh_0 + E[rs]V + E[r](E[rs])^2)^2}
\]

\[
= \frac{\sigma_0^2(E[rs]V + E[r](E[rs])^2)E[r]Vh_0}{(E[r]Vh_0 + E[rs]V + E[r](E[rs])^2)^2} = \frac{E[r]V(E[rs]V + E[r](E[rs])^2)}{(E[r]Vh_0 + E[rs]V + E[r](E[rs])^2)^2}
\]
The last equality uses the fact that \( h_0 \sigma_0^2 = 1 \), as follows from (3).

Now we look at \( \gamma^2 V \). Using (7) yields

\[
\gamma^2 V = \frac{V(V + E[r]E[rs])^2}{(E[r]Vh_0 + E[rs]V + E[r](E[rs])^2)^2}.
\]

Finally, combining the two terms we get

\[
\beta (\sigma_0^2 - \beta \sigma_0^2) \gamma^2 V = \frac{E[r]V(E[rs]V + E[r](E[rs])^2) - V(V + E[r]E[rs])^2}{(E[r]Vh_0 + E[rs]V + E[r](E[rs])^2)^2} - \frac{V^2 + E[r]E[rs]V^2}{(E[r]Vh_0 + E[rs]V + E[r](E[rs])^2)^2} < 0.
\]

Q.E.D.

Proof of Proposition 4.1.

The market clearing condition is

\[
\frac{1}{2\delta} \ln \frac{\bar{\varphi} + \delta}{\bar{\varphi} - \delta} = \int_{\bar{\varphi} - \delta}^{\bar{\varphi} + \delta} \frac{1}{a\varphi_i\sigma^2} (\mu - \phi_i - P) \frac{1}{2\delta} d\phi_i
\]

Computing the integral on the right-hand side yields:

\[
\int_{\bar{\varphi} - \delta}^{\bar{\varphi} + \delta} \frac{1}{a\varphi_i\sigma^2} (\mu - \phi_i - P) \frac{1}{2\delta} d\phi_i = \frac{1}{2\delta} \frac{\mu - P}{a\sigma^2} \ln \frac{\bar{\varphi} + \delta}{\bar{\varphi} - \delta} - \frac{1}{a\sigma^2}
\]

Plugging this into (31) and dividing both sides by \( \frac{1}{2\delta} \ln \left[ (\bar{\varphi} + \delta)/(\bar{\varphi} - \delta) \right] \) gives

\[
1 = \frac{\mu - P}{a\sigma^2} - \frac{1}{a\sigma^2} \ln \left[ (\bar{\varphi} + \delta)/(\bar{\varphi} - \delta) \right].
\]

Finally, multiplying both sides by \( a\sigma^2 \), rearranging, and using the expression for \( \bar{x} \) yields:

\[
\mu - P = a\sigma^2 \frac{1}{\bar{x}}.
\]

Q.E.D.
Proof of Proposition 4.2.
The Proof is the same as that of Proposition 4.1, when instead of $\mu - P$ we need to use $\phi^*$, and the upper limit of integration of individuals demands should now be $\phi^*$ instead of $\phi + \delta$.

Q.E.D.

Proof of Proposition 4.3.
From (28), it follows that
\[
\frac{d\pi}{dk} = \frac{d}{dk} \left[ \frac{\phi^*}{2\delta} \right] = \frac{2\delta \phi^*}{4\delta^2} - \phi^* \frac{2\delta}{4\delta^2}.
\]
Here we used the fact that the numerator and the denominator in $(\phi - \delta)/(2\delta)$ are both proportional to $k$ and so the ratio is not affected when $k$ varies.

Denote by $F(\phi^*, k)$ the right-hand side of (26). We also need to replace $\bar{\phi}$ and $\delta$ by $\bar{\phi}/k$ and $\delta/k$, respectively, to reflect how the model’s parameters change when the investors’ endowments are multiplied by $k$. We have
\[
F(\phi^*, k) = \frac{\phi^*}{a\sigma^2} \ln \frac{k\phi^*}{\phi - \delta} - \frac{1}{a\sigma^2} \left( \phi^* - \frac{\phi - \delta}{k} \right).
\]
To differentiate the implicit function $\phi^*(k)$, we need to compute $F_{\phi^*}$ and $F_k$. We have
\[
\frac{dF}{d\phi^*} = \frac{1}{a\sigma^2} \ln \frac{k\phi^*}{\phi - \delta} + \frac{\phi^*}{a\sigma^2} \frac{1}{\phi^*} - \frac{1}{a\sigma^2} = \frac{1}{a\sigma^2} \ln \frac{k\phi^*}{\phi - \delta},
\]
and
\[
\frac{dF}{dk} = \frac{\phi^*}{a\sigma^2} \frac{1}{k} - \frac{1}{a\sigma^2} \frac{\phi - \delta}{k^2}.
\]
We now have that
\[
\frac{d\phi^*}{dk} = -\frac{dF/dK}{dF/d\phi^*} = \frac{\phi^* - \phi - \delta}{\ln \frac{k\phi^*}{\phi - \delta}}.
\]
Plugging this into (32) and ignoring the denominator as we are only interested in the sign of $d\pi/dk$, we get
\[
2\delta \frac{\phi^* - \phi - \delta}{\ln \frac{k\phi^*}{\phi - \delta}} + \phi^* \frac{2\delta}{k^2}.
\]

37
We now multiply the RHS by
\[ \frac{k^2}{2\delta a\sigma^2} \ln \frac{k \phi^*}{\phi - \delta} \]
which, being positive, does not change the sign of \( d\pi/dk \). This yields
\[ \frac{\phi^*}{a\sigma^2} \ln \frac{k \phi^*}{\phi - \delta} - \frac{k}{a\sigma^2} \left( \frac{\phi^*}{k} - \frac{\bar{\phi} - \delta}{k} \right) . \]
Evaluating the last expression at \( k = 1 \), we see that it equals the right-hand side in (26) and so is positive since the left-hand side is positive. Hence, an infinitesimal increase (decrease) in endowments increases (decreases) the market participation. But since this is true for any \( \phi^* \) it means that \( \phi^*(k) \) increases for all \( k \)-s until the full participation is achieved.

Q.E.D.

Proof of Proposition 4.4.
The result immediately follows from (33) because \( \phi^* > \bar{\phi} - \delta \).

Q.E.D.
References


